Flexural - Torsional Buckling Analysis of Thin Walled Columns Using the Fourier series Method

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Abstract**—** *In this work, the governing differential equations of elastic column buckling represented by a system of three coupled differential equations in the three unknown displacement functions, v(x), w(x) and θ(x) are solved using the method of Fourier series. The column was pinned at both ends x = 0, x = l. The unknown displacements were assumed to be a Fourier sine series of infinite terms, which was found to satisfy apriori the pinned conditions at the ends and substituted into the governing equations. The governing equations were found to reduce to a system of algebraic eigenvalue – eigenvector problem. The buckling equation was found to be a cubic polynomial for the general asymmetric sectioned column. The buckling modes were found as flexural torsional buckling modes. For columns with monosymmetric sections, it was found that the buckling mode could be flexural or flexural – torsional depending on the root of the cubic polynomial buckling equation which is the smallest. For columns with bisymmetric sections, it was found that the buckling modes are uncoupled and bisymmetric columns could fail by pure flexural buckling about the axes of symmetry or pore torsional buckling. The findings are in excellent agreement with Timoshenko's solutions.*

Keywords—Monosymmetric columns, bisymmetric columns, flexural torsional buckling mode, algebraic-eigen vector problem.

I. INTRODUCTION/LITERATURE REVIEW

Euler's [1] work on column flexural buckling presented the first analytical method of determining the buckling strengths of slender columns. St Venant [2] later worked on uniform torsion and presented the first reliable work on the twisting response of structures to torsion. Flexural torsional buckling was studied by Michell [3] and also by Prandtl [4] who considered the lateral buckling of beams of narrow rectangular cross sections.

Prandtl and Michell's work were extended by Timoshenko [5] to include the effects of warping torsion in **T** section beams. The works of Wagner [6], Vlasov[7], Timoshenko [8] and Timoshenko [9] led to the development of a general theory of flexural torsional buckling. Other researchers who have studied flexural torsional buckling of structures include Nwakali[10], Timoshenko and Gere [11], Alsayed[12],Trahair[13], Zlatko[14] and Al-Sheikh [15].

II. THEORETICAL FRAMEWORK

The generalized elastic column buckling problem formulated in terms of the displaced configuration is represented by the following set of three linear differential equations obtained when the non linear terms are neglected [16][13].

$$E(I_{yy}I_{zz}-I_{yz}^2)\frac{d^4v}{dx^4}-I_{yy}N_x\frac{d^2v}{dx^2}+I_{yz}N_x\frac{d^2w}{dx^2}+\Big[I_{yy}M_{ly}+I_{yz}M_{lz}-$$

$$N_x(e_zI_{yy}+e_yI_{yz}\Big]\frac{d^2\theta}{dx^2}+(I_{yy}V_z+I_{yz}V_y)\frac{d\theta}{dx}=q_yI_{yy}-q_zI_{yz} \qquad (2.1)$$

$$E(I_{yy}I_{zz}-I_{yz}^2)\frac{d^4w}{dx^4}-I_{zz}N_y\frac{d^2w}{dx^2}+I_{yz}N_x\frac{d^2v}{dx^2}+\Big[-I_{zz}M_{lz}-I_{yz}M_{ly}+$$

$$N_x(e_yI_{zz}+e_zI_{yz})\Big]\frac{d^2\theta}{dx^2}-(I_{zz}V_y-I_{yz}V_z)\frac{d\theta}{dx}=q_zI_{zz}-q_yI_{yz} \qquad (2.2)$$

$$EC_w\frac{d^4\theta}{dx^4}-\left(GJ+\frac{I_EN_x}{A}+C_zM_{lz}+C_yM_{ly}+\frac{H_wW_w}{C_w}\right)\frac{d^2\theta}{dx^2}+$$

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$$(M_{ly} - e_z N_x)\frac{d^2 v}{dx_x^2} - (M_{lz} - e_y N_x)\frac{d^2 w}{dx_x^2} + V_z \frac{dv}{dx} - V_y \frac{dw}{dx} -$$

$$\left(C_z V_y + C_y V_z + \frac{H_w V_w}{C_w} \right)\frac{d\theta}{dx} = t(x) \qquad (2.3)$$

where $M_{lz}, M_{ly} =$ moments caused by transverse loads only.

$I_E = I_{yy} + I_{zz} + (e_y^2 + e_z^2)A =$ polar moment of inertia about the shear centre.

$$H_y = \iint_A z(y^2 + z^2)dA \qquad I_{yy} = \int_A z^2 dA$$

$$H_z = \iint_A y(y^2 + z^2)dA \qquad I_{yy} = \int_A y^2 dA$$

$$H_w = \iint_A 2(w_0 - w)(y^2 + z^2)dA \quad I_{yz} = \int_A yz\, dA$$

$$C_y = \frac{I_{zz}H_y - I_{yz}H_z}{I_{yy}I_{zz} - I_{yz}^2} = 2e_z$$

$$C_z = \frac{I_{yy}H_z - I_{yz}H_y}{I_{yy}I_{zz} - I_{yz}^2} = 2e_y$$

J = St Venant torsional stiffness of the section
E = Young's modulus of elasticity
G = shear modulus
I_{yy}, I_{zz} = moments of inertia
C_w = warping constant
I_{yz} = product of inertia
e_y, e_z = coordinates of the shear center
V_y, V_z = shear forces
q_y, q_z = transverse loads
N_x = axial load
$v(x)$,$w(x)$ are transverse displacements
θ_x = twist rotational displacement
x = longitudinal axial coordinate, yz is the plane of the cross section
A = area of the cross section of the column.

The governing equilibrium equations for the generalized column buckling problem is represented by the system of three simultaneous differential equations in the three unknown displacements $v(x)$, $w(x)$ and $\theta(x)$. For columns with prismatic cross-sections, the elasticity properties (GJ and E) as well as the inertial and geometrical properties are constant; but the load coefficients (M_{ly}, M_{lz}, V_y, V_z, W_w, V_w) are variables depending on the axial longitudinal coordinate (x). The system of governing differential equations thus have variable coefficients, rendering them difficult to solve mathematically.

However, simplifications of the system of governing differential equations can be obtained by using special characteristics of different types of column problems. A simplification of the governing equilibrium equations can be obtained if the yz coordinates are principal coordinates. Then $I_{yz} = 0$, and the governing differential equations become [16][13]

$$EI_{zz}\frac{d^4 v}{dx^4} - N_x \frac{d^2 v}{dx^2} + (M_{ly} - N_x e_z)\frac{d^2\theta}{dx^2} + V_{zy}\frac{d\theta}{dx} = q_y \qquad (2.4)$$

$$EI_{yy}\frac{d^4 w}{dx^4} - N_x \frac{d^2 w}{dx^2} + (-M_{lz} + N_x e_y)\frac{d^2\theta}{dx^2} - V_y \frac{d\theta}{dx} = q_z \qquad (2.5)$$

$$EC_w \frac{d^4 w}{dx^4} - \left(GJ + \frac{I_E N_x}{A} + C_z M_{lz} + C_y M_{ly} + \frac{H_w W_w}{C_w} \right)\frac{d^2\theta}{dx^2} + (M_{ly} - e_z N_x)\frac{d^2 v}{dx^2}$$

$$-(M_{lz} - e_y N_x)\frac{d^2 w}{dx^2} + V_z \frac{dv}{dx} - V_y \frac{dw}{dx} - \left(C_z V_y + C_y V_z + \frac{H_w V_w}{C_w} \right)\frac{d\theta}{dx} = t(x) \quad (2.6)$$

If the member is free of transverse loads q_y, q_z become zero; and the transverse moments are constant, then the shear forces V_y, V_z are also zero. For member (columns) subjected only to axial compressive load N_x acting through the centroid of the section, the moments due to the transverse loads vanish, the applied torque vanishes; and if the load is applied such that the bimoment vanishes, the system of differential equations become [15][16]

$$EI_{zz}\frac{d^4 v}{dx^4} + N_x \frac{d^2 v}{dx^2} + N_x e_z \frac{d^2\theta}{dx^2} = 0 \quad (2.7)$$

$$EI_{yy}\frac{d^4 w}{dx^4} + N_x \frac{d^2 w}{dx^2} - N_x e_y \frac{d^2\theta}{dx^2} = 0 \quad (2.8)$$

$$EC_w \frac{d^4\theta}{dx^4} - \left(GJ - \frac{I_E N_x}{A} \right)\frac{d^2\theta}{dx^2} + N_x e_z \frac{d^2 v}{dx^2} - N_x e_y \frac{d^2 w}{dx^2} = 0 \quad (2.9)$$

The system of differential Equations (2.7), (2.8) and (2.9) represent the governing equilibrium equations for a particular case of the generalized elastic column buckling problem presented by Equations (2.1), (2.2) and (2.3) when $I_{yz} = 0$, $q_y = q_z = 0$, $V_y = \theta_z = 0$, $M_{ly} = M_{lz} = 0$, $W_w = 0$.

III. APPLICATION OF THE FOURIER SERIES METHOD

We seek to apply the Fourier series method to solve the system of differential Equations (2.7), (2.8) and (2.9) representing the governing-equilibrium equations for the elastic column buckling problem for the case when the ends at $x = 0$, $x = l$are on pinned supports.

For pinned ends at $x = 0$, $x = l$, the relevant boundary conditions are

$$\begin{aligned}
&v(x=0) = 0 \qquad v''(x=0) = 0 \\
&w(x=0) = 0 \qquad w''(x=0) = 0 \\
&\theta(x=0) = 0 \qquad \theta''(x=0) = 0 \\
&v(x=l) = 0 \qquad v''(x=l) = 0 \\
&w(x=l) = w''(x=l) = 0 \\
&\theta(x=l) = \theta''(x=l) = 0
\end{aligned} \quad (3.1)$$

Suitable Fourier series representations of the three unknown displacement functions that satisfy apriori the above boundary conditions are given by

$$\begin{aligned}
v(x) &= \sum_{m=1}^{\infty} v_m \sin\frac{m\pi x}{l} \\
w(x) &= \sum_{m=1}^{\infty} w_m \sin\frac{m\pi x}{l} \\
\theta(x) &= \sum_{m=1}^{\infty} \theta_m \sin\frac{m\pi x}{l}
\end{aligned} \quad (3.2)$$

where v_m, w_m and θ_m are infinite number of unknown coefficients of the Fourier sine series representations of the unknown displacement functions $v(x)$, $w(x)$ and $\theta(x)$ that we seek to determine. If Equation (3.2) represent solutions of the governing Equations (2.7), (2.8) and (2.9), then

$$EI_{zz}\frac{d^4}{dx^4}\left(\sum_{m=1}^{\infty} v_m \sin\frac{m\pi x}{l} \right) + N_x \frac{d^2}{dx^2}\left(\sum_{m=1}^{\infty} v_m \sin\frac{m\pi x}{l} \right) + N_x e_z \frac{d^2}{dx^2}\left(\sum_{m=1}^{\infty} \theta_m \sin\frac{m\pi x}{l} \right) = 0 \quad (3.3)$$

$$EI_{yy}\frac{d^4}{dx^4}\left(\sum_{m=1}^{\infty} w_m \sin\frac{m\pi x}{l} \right) + N_x \frac{d^2}{dx^2}\left(\sum_{m=1}^{\infty} w_m \sin\frac{m\pi x}{l} \right) - N_x e_y \frac{d^2}{dx^2}\left(\sum_{m=1}^{\infty} \theta_m \sin\frac{m\pi x}{l} \right) = 0 \quad (3.4)$$

$$\begin{aligned}
&EC_w \frac{d^4}{dx^4}\left(\sum_{m=1}^{\infty} \theta_m \sin\frac{m\pi x}{l} \right) - \left(GJ - \frac{I_E N_x}{A} \right)\frac{d^2}{dx^2}\left(\sum_{m=1}^{\infty} \theta_m \sin\frac{m\pi x}{l} \right) \\
&\quad + N_x e_z \frac{d^2}{dx^2}\left(\sum_{m=1}^{\infty} v_m \sin\frac{m\pi x}{l} \right) - N_x e_y \frac{d^2}{dx^2}\left(\sum_{m=1}^{\infty} w_m \sin\frac{m\pi x}{l} \right)
\end{aligned} \quad (3.5)$$

$$EI_{zz}\sum_{m=1}^{\infty}\left(\frac{m\pi}{l}\right)^4 v_m \sin\frac{m\pi x}{l} - N_x\sum_{m=1}^{\infty}\left(\frac{m\pi}{l}\right)^2 v_m \sin\frac{m\pi x}{l} - N_x e_z\sum_{m=1}^{\infty}\left(\frac{m\pi}{l}\right)^2 v_m \sin\frac{m\pi x}{l} = 0 \quad (3.6)$$

$$EI_{yy}\sum_{m=1}^{\infty}\left(\frac{m\pi}{l}\right)^4 w_m \sin\frac{m\pi x}{l} - N_x\sum_{m=1}^{\infty}\left(\frac{m\pi}{l}\right)^2 w_m \sin\frac{m\pi x}{l} + N_x e_y\sum_{m=1}^{\infty}\left(\frac{m\pi}{l}\right)^2 \theta_m \sin\frac{m\pi x}{l} = 0 \quad (3.7)$$

$$EC_w\sum_{m=1}^{\infty}\left(\frac{m\pi}{l}\right)^4 \theta_m \sin\frac{m\pi x}{l} + \left(GJ - \frac{I_E N_x}{A}\right)\sum_{m=1}^{\infty}\left(\frac{m\pi}{l}\right)^2 \theta_m \sin\frac{m\pi x}{l}$$

$$- N_x e_z\sum_{m=1}^{\infty}\left(\frac{m\pi}{l}\right)^2 v_m \sin\frac{m\pi x}{l} + N_x e_y\sum_{m=1}^{\infty}\left(\frac{m\pi}{l}\right) w_m \sin\frac{m\pi x}{l} = 0 \quad (3.8)$$

Simplifying,

$$\sum_{m=1}^{\infty}\left\{\left(EI_{zz}\left(\frac{m\pi}{l}\right)^2 - N_x\right)v_m \sin\frac{m\pi x}{l} - N_x e_z \theta_m \sin\frac{m\pi x}{l}\right\} = 0 \quad (3.9)$$

$$\sum_{m=1}^{\infty}\left\{\left(EI_{yy}\left(\frac{m\pi}{l}\right)^2 - N_x\right)w_m \sin\frac{m\pi x}{l} + N_x e_y \theta_m \sin\frac{m\pi x}{l}\right\} = 0 \quad (3.10)$$

$$\sum_{m=1}^{\infty}\left\{\left(EI_w\left(\frac{m\pi}{l}\right)^2 + \left(GJ - \frac{I_E N_x}{A}\right)\right)\theta_m \sin\frac{m\pi x}{l} - N_x e_z v_m \sin\frac{m\pi x}{l} + N_x e_y w_m \sin\frac{m\pi x}{l}\right\} = 0 \quad (3.11)$$

In matrix format, we have

$$\sum_{m=1}^{\infty}\left\{\begin{matrix} \left(EI_{zz}\left(\frac{m\pi}{l}\right)^2 - N_x\right) & 0 & -N_x e_z \\ 0 & \left(EI_{yy}\left(\frac{m\pi}{l}\right)^2 - N_x\right) & N_x e_y \\ -N_x e_z & N_x e_y & \left(EC_w\left(\frac{m\pi}{l}\right)^2 + \left(GJ - \frac{I_E N_x}{A}\right)\right) \end{matrix}\right\} \times \begin{pmatrix} v_m \\ w_m \\ \theta_m \end{pmatrix} \sin\frac{m\pi x}{l} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (3.12)$$

The stability problem represented by the system of differential equations is now reduced to an algebraic eigenvector eigenvalue problem given by the homogeneous Equation (3.12). For non-trivial solutions, the characteristic buckling equation is the determinatal Equation (3.13)

$$\begin{vmatrix} \left(EI_{zz}\left(\frac{m\pi}{l}\right)^2 - N_x\right) & 0 & -N_x e_z \\ 0 & \left(EI_{yy}\left(\frac{m\pi}{l}\right)^2 - N_x\right) & N_x e_y \\ -N_x e_z & N_x e_y & \left(EC_w\left(\frac{m\pi}{l}\right)^2 + GJ - \frac{I_E N_x}{A}\right) \end{vmatrix} = 0 \quad (3.13)$$

Let $$P_{zz} = EI_{zz}\left(\frac{m\pi}{l}\right)^2$$

$$P_{yy} = EI_{yy}\left(\frac{m\pi}{l}\right)^2$$

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$$P_\phi = \frac{A}{I_E}\left(EC_w\left(\frac{m\pi}{l}\right)^2 + GJ\right)$$

$$\therefore \qquad EC_w\left(\frac{m\pi}{l}\right)^2 + GJ = \frac{P_\phi I_E}{A}$$

Hence we have

$$\begin{vmatrix} (P_{zz}-N_x) & 0 & -N_x e_z \\ 0 & (P_{yy}-N_x) & N_x e_y \\ -N_x e_z & N_x e_y & \left(\frac{P_\phi I_E}{A} - \frac{I_E N_x}{A}\right) \end{vmatrix} = 0 \qquad (3.14)$$

By expansion of the determinant, we have

$$(P_{zz}-N_x)\begin{vmatrix} (P_{yy}-N_x) & N_x e_y \\ N_x e_y & \frac{I_E}{A}(P_\phi - N_x) \end{vmatrix} - N_x e_z \begin{vmatrix} 0 & (P_{yy}-N_x) \\ -N_x e_z & N_x e_y \end{vmatrix} = 0 \qquad (3.15)$$

$$(P_{zz}-N_x)\left[(P_{yy}-N_x)(P_\phi - N_x)\frac{I_E}{A} - (N_x e_y)^2\right] - N_x e_z(0 + N_x e_z(P_{yy}-N_x)) = 0 \qquad (3.16)$$

$$(P_{zz}-N_x)(P_{yy}-N_x)(P_x - N_x)\frac{I_E}{A} - (N_x e_y)^2(P_{zz}N_x) - (N_x e_z)^2(P_{yy}-N_x) = 0 \qquad (3.17)$$

$$(P_{zz}-N_x)(P_{yy}-N_x)(P_\phi - N_x) - N_x^2 e_y^2 \frac{A}{I_E}(P_{zz}-N_x) - N_x^2 e_z^2 \frac{A}{I_E}(P_{yy}-N_x) = 0 \qquad (3.18)$$

But

$$r_0^2 = e_y^2 + e_z^2 + \left(\frac{I_{xx}+I_{yy}}{A}\right)$$

$$I_E = I_{xx} + I_{yy} + (e_x^2 + e_z^2)A$$

$$\frac{I_E}{A} = \frac{I_{xx}+I_{yy}}{A} + e_y^2 + e_z^2 = r_0^2$$

Hence,

$$(P_{zz}-N_x)(P_{yy}-N_x)(P_\phi - N_x) - N_x^2\frac{e_y^2}{r_0^2}(P_{zz}-N_x) - N_x^2\frac{e_z^2}{r_0^2}(P_{yy}-N_x) = 0 \qquad (3.19)$$

$$(P_{zz}-N_x)(P_{yy}-N_x)(P_\phi - N_x) - N_x^2\frac{e_y^2}{r_0^2}(P_{zz}-N_x) - \frac{N_x^2 e_z^2}{r_0^2}(P_{yy}-N_x) = 0 \qquad (3.20)$$

This is the characteristic equation for determining the buckling load of a column with an asymmetric cross-section. The buckling equation is a third order polynomial in N_x. Thus, it can be solved using the methods for solving polynomials to obtain the three roots P_{cr_1}, P_{cr_2} and P_{cr_3}. The smallest of the three critical buckling loads will govern the buckling behaviour of the column.
However, if the *zz* axis is the axis of symmetry of the cross-section, $e_y = 0$, and the characteristic buckling equation simplifies to become

$$(P_{zz}-N_x)(P_{yy}-N_x)(P_\phi - N_x) - N_x^2\frac{e_z^2}{r_0^2}(P_{yy}-N_x) = 0$$

…(3.21)

or,

$$(P_{yy}-N_x)\left[(P_{zz}-N_x)(P_\phi - N_x) - \frac{N_x^2 e_z^2}{r_0^2}\right] = 0$$

…(3.22)

Hence, in this case (monosymmetric cross-section), the column can buckle in two possible buckling modes namely pure flexural buckling in the *yy* – axis, and flexural torsional buckling, otherwise. If P_{yy} is the smallest of the three roots, of Equation (3.22) the monosymmetric section column will buckle in pure Eulerian flexure, otherwise flexural torsional buckling failure will take place.

If the column has two axis of symmetry then $e_y = 0$, $e_y = 0$ and the characteristic buckling equation simplifies to

$$(P_{zz} - N_x)(P_{yy} - N_x)(P_{\phi} - N_x) = 0 \qquad (3.23)$$

We find that for bisymmetric columns, the buckling equations become uncoupled and the three roots become P_{zz}, P_{yy} and P_{ϕ}. For bisymmetric columns, the column can buckle in pure Eulerian flexure in the *yy* or *zz* axes, or pure torsional buckling mode. The smallest of the three roots governs the buckling mode. Critical values of each buckling mode is obtained in each case when $m = 1$.

IV. DISCUSSIONS AND CONCLUSIONS

The system of governing differential equations of buckling of an elastic column with thin walled sections have been solved for pinned pinned columns using the Fourier series method. In general, the equations were found to be reduced to a system of algebraic eigenvalue eigenvector problem. For the general use of asymmetric columns, the characteristic buckling equation was found to be identical with the Timoshenko's solution for the same problem. The buckling equation for asymmetric columns was found as Equation (3.20). The equation is a cubic polynomial in N_x as the unknown. The solution will yield three values for N_x called the three critical loads as P_{cr_1}, P_{cr_2} and P_{cr_3}. All the buckling modes in this case are flexural – torsional buckling modes. The critical buckling load, which is the lowest of the three loads P_{cr_1}, P_{cr_2} and P_{cr_3} will always be smaller than the Euler critical flexural buckling loads P_{Ezz}, P_{Eyy} and the pure torsional buckling load P_{ϕ}. For columns with asymmetric sections the buckling mode always include all three displacements $v(x)$, $w(x)$ and $\theta(x)$; hence it is a flexural – torsional buckling mode.

For columns with monosymmetric cross-sections, one of the governing differential equations will become uncoupled. This results in two possible modes of buckling failure – pure flexural buckling failure in the *yy* axes (for columns symmetrical about the *zz* axes) and flexural torsional buckling. The characteristic buckling equation was found as Equation (3.22). For columns with two axes of symmetry, the governing differential equations of equilibrium become uncoupled. The characteristic buckling equation was found as Equation (3.23). The buckling equations are uncoupled and the three roots indicate the bisymmetric column can buckle in pure Eulerian flexure in the axes of symmetry or pure torsional buckling mode.

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